Directions: You have 120 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator.

Note: For the last two problems, please only pick one to do. Indicate the one that you do not wish to be graded on the following table.

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(1) (20 points) Fill in the blanks:

(a) Let \( X_1, X_2, \ldots, X_9 \) be a random sample from the normal distribution \( N(-5, 9) \). Find \( a \) such that
\[ P(\overline{X} > a) = 0.95. \]

Answer: \( a = \) _______________.

(b) Let \( X_1, X_2, \ldots, X_{15} \) be a random sample from the normal distribution \( N(30, 100) \). Find \( b \) such that
\[ P(\sum_{i=1}^{15}(X_i - \overline{X})^2 \leq b) = 0.99. \]

Answer: \( b = \) _______________.

(c) In a public opinion poll for a close presidential election, let \( p \) denote the proportion of voters who favor candidate A. How large a sample should be taken if we want the maximum error of the estimate of \( p \) to be equal to 0.03 with 95% confidence?

Answer: ____________.

(d) Let \( X \) equal excess weight of soap in a 1000-gram bottle. Assume that the distribution of \( X \) is \( N(\mu, 169) \). What sample size is required so that we have 95% confidence that the maximum error of the estimate of \( \mu \) is 1.5?

Answer: ____________
(2) (15 points) A test was conducted to determine if a wedge on the end of a plug fitting designed to hold a seal onto the plug was doing its job. The data taken were in form of measurements of the force required to remove a seal from the plug first with the wedge in place, say, $X$, and then the force required without the wedge, say, $Y$. Assume that the distribution of $X$ and $Y$ are $N(\mu_X, \sigma^2)$ and $N(\mu_Y, \sigma^2)$, where the 2 distributions have the equal variance $\sigma^2$.

Six independent observations of $X$ are:

$3.26, \ 2.26, \ 2.62, \ 2.36, \ 3.00, \ 2.62 \ (\bar{x} = 2.69, \ s_x = 0.38)$

Six independent observations of $Y$ are:

$1.90, \ 1.46, \ 1.54, \ 1.32, \ 1.56, \ 2.00 \ (\bar{y} = 1.63, \ s_y = 0.26)$

(a) Find a 95% confidence interval for $\mu_X - \mu_Y$.

(b) Is the wedge necessary? Explain briefly.
(3) (15 points) Let \( X \) equal the forced vital capacity (FVC) in liters for a female college student. Assume that the distribution of \( X \) is approximately \( N(\mu, \sigma^2) \). Suppose it is known that \( \mu = 3.4 \) liters. A volleyball coach claims that the FVC of volleyball players is greater than 3.4 liters. She plans to test \( H_0 : \mu = 3.4 \, \text{vs} \, H_1 : \mu > 3.4 \) using a random sample of size \( n = 9 \).

(a) Define a rejection region for which \( \alpha = 0.05 \).

(b) Given that the random sample yielded the following FVC:

\[
3.4 \ 3.6 \ 3.8 \ 3.3 \ 3.4 \ 3.5 \ 3.6 \ 3.7 \ 3.7 \ (\bar{x} = 3.556, \ s = 0.167)
\]

What is your conclusion?

(c) What is the approximate \( p \)-value of this test?
(4) (15 points) A company manufactures machines to package soap powder. The mean and variance of a sample of eight 3-ounce boxes were found to be 3.15 and 0.02, respectively.

(a) Test the hypothesis that the variance of the population of weight measurements is $\sigma^2 = 0.01$ against the alternative $\sigma^2 > 0.01$. Please state the test statistics, rejection region, and your conclusion clearly. Use an $\alpha = 0.05$ level of significance.

(b) What assumptions are required for this test you used?
(5) (15 points) The proportion of adults living in a small town who are college graduates is claimed to be \( p = 0.4 \). To test this claim, a random sample of 15 adults is selected. If the number of college graduates in our sample is anywhere from 4 to 8, we will accept the null hypothesis that \( p = 0.4 \); otherwise, we shall conclude that \( p \neq 0.4 \).

(a) Use the binomial distribution to evaluate the type I error \( \alpha \).

(b) Evaluate type II error \( \beta \) for the alternative hypothesis \( p = 0.15 \).

(c) Suppose 200 adults are selected and the acceptance region for \( H_0 \) is defined to be \( 70 \leq x \leq 90 \), where \( x \) is the number of college graduates in the sample. Please use normal distribution to approximate the type II error \( \beta \) for the alternative hypothesis \( p = 0.15 \).
(6) (10 points) A random sample of size 36 is taken from the distribution with p.d.f.
\[ f(x) = 1 - \frac{x}{2}, \quad 0 \leq x \leq 2. \]

(a) Find \( \mu \) and \( \sigma^2 \) of this distribution.

(b) Find, approximately, \( P(2/3 < \bar{X} < 5/6) \).
You only need to do one of the following 2 problems. Please cross out (on the cover page) the problem that you don’t wish to be graded, either Problem 7 or Problem 8.

(7) (10 points) Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from the uniform distribution on the interval \([\theta, 1]\), i.e., with p.d.f.

\[
f(x; \theta) = \begin{cases} 
\frac{1}{1-\theta} & \theta \leq x \leq 1; \\
0 & \text{elsewhere}.
\end{cases}
\]

(a) Find the maximum likelihood estimator of \( \theta \).

(b) (Numerical Application) Based on the following sample, give a point estimate of \( \theta \).

0.77, 0.30, 0.90, 0.45, 0.62.
(8) (10 points) Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n$ from a distribution with p.d.f.

$$f(x; \theta) = \begin{cases} 
\left(\frac{2}{\theta^2}\right) (\theta - x), & \text{for } 0 \leq x \leq \theta, \\
0, & \text{otherwise.}
\end{cases}$$

Find the method of moments estimator for $\theta$. 
Answers:

(1) Problem 1:
   (a) -6.645
   (b) 2914
   (c) 1068
   (d) 289

(2) Problem 2:
   (a) (0.64, 1.48)
   (b) Yes

(3) Problem 3:
   (a) $C = \{ t(8) > 1.86 \} = \{ \bar{X} > 3.4 + 0.62S \}$
   (b) $t_{obs} = 2.80$, Reject $H_0$
   (c) $p$-value $\approx 0.01$

(4) Problem 4:
   (a) Can’t reject $H_0$.
   (b) The underlying distribution is normal.

(5) Problem 5:
   (a) 0.186
   (b) 0.177
   (c) $\approx 0$

(6) Problem 6:
   (a) $\mu = 2/3, \sigma^2 = 2/9$
   (b) 0.483

(7) Problem 7:
   (a) $\hat{\theta}_{mle} = X_{(1)}$, the minimum of all $X_1, X_2, \ldots, X_n$.
   (b) 0.3

(8) Problem 8:
   $\hat{\theta}_{mom} = 3\bar{X}$