Polytechnic University

MA 2122 FINAL AUGUST 25, 2003

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have 90 minutes to answer the following questions. You must show all your work as clearly as possible. You may use a calculator, but you must give the exact value for each of the numerical answers. The last three pages contain a list of useful formulas. You may tear these pages out.

Problem	Possible	Points
1	10	
2	10	
3	15	
4	15	
5	10	
6	10	
7	15	
8	15	
Total	100	

(1) (Worksheet I, II) In each of the following cases, you must set up an iterated integral, but you are not asked to evaluate it.

(a) Reversing the order of integration, the integral $\int_0^1 \int_{e^y}^{e} \frac{x}{\ln(x)}\, dx\, dy$ becomes

(b) Let R be the region in the first quadrant of the xy-plane bounded by the curves with equations

$$x^2 + y^2 = 4, \quad y = x\sqrt{3} \quad \text{and } y\sqrt{3} = x.$$

An iterated integral, in **polar coordinates**, equal to the area of R is

(2) (Worksheet I, II) In each of the following cases, you must set up an iterated integral in the coordinates of your choice, but you are not asked to evaluate it.

(a) In a particular spherical cloud of gas of radius 3, the mass density at each point is equal to the distance from that point to the surface of the cloud.

A triple integral representing the total mass of the cloud of gas is

(b) Two spheres, each of radius 1, have centers that are 1 unit apart. Let W be the region between the two spheres.

A triple integral representing the volume of W is

(Hint: *You may assume that one sphere is centered at the origin and the second one is centered at* $(0, 0, 1)$.)

(3) (Worksheet III) Consider the vector field

$$\vec{F}(x,y) = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}.$$

Calculate the line integral of $\vec{F}$ along the square going from $(a,-a)$ to (a,a) to $(-a,a)$ to $(-a,-a)$, then back to $(a,-a)$. Here a is a positive constant.

(4) (Worksheet I) An infinite cylinder centered on the z-axis, with base of radius 1, is cut by the planes $z = -2$ and $z = x$. Let W be the solid bounded by the cylinder and the two planes.

Assuming W has constant density, find the z-coordinate of its center of mass.

(5) (Worksheet III, IV) Fill in the blanks.

(a) A potential function for the vector field

$$\vec{F}(x, y, z) = (2y\cos(2x) - \sin(2y))\,\vec{i} + (\sin(2x) - 2x\cos(2y))\,\vec{j}$$

is

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(b) If $\vec{F}$ is the vector field from part (a) and C is some continuous curve from $(\pi/3, 2\pi/3)$ to $(2\pi/3, \pi/3)$, then

$\int_C \vec{F} \cdot d\vec{r} =$ ________________________________

(6) (Worksheet III, IV) Fill in the blanks.

(a) Let T be the triangle of vertices $(-1,0)$, $(1,0)$ and $(0,1)$ with counterclockwise orientation. If

$$\vec{F}(x,y) = \left(x^2 e^y\right)\,\vec{i} + \left(x^2 e^x\right)\,\vec{j},$$

then by Green's theorem

$$\int_T \vec{F}\cdot d\vec{r} = \underline{\hspace{6cm}}$$

Write the resulting iterated integral. You do not need to evaluate the integral.

(b) A smooth vector field G has $\operatorname{curl}\vec{G}(x,y,z) = (2+x)\vec{i} - (3+y)\vec{j} + (5+z)\vec{k}$. The circulation of $\vec{G}$ around a circle of radius 1 in the xy-plane, oriented counterclockwise when viewed from the positive z-axis, is equal to

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(7) (Worksheet IV) Compute the flux of the vector field $\vec{F}(\vec{r}) = \vec{r}$, through the **closed** surface

$$z = 2 - (x^2 + y^2), \quad z \geq 0,$$

oriented outward.

(8) (Page 871, Problem 23; Worksheet IV) The flux of the constant vector field $a\vec{i}+b\vec{j}+c\vec{k}$ through a square of side 2 in the plane $x = 5$, oriented in the positive x-direction, is 24.

The flux of the same vector field through a square of side 3 in the plane of normal $\vec{n} = \vec{j}+\vec{k}$ is 27, while its flux through a square of side 4 in the plane $z = 3$, oriented in the negative z-direction, is 32.

Then $a =$ _______ , $b =$ _______ and $c =$ _______ .

(*You must show your work.*)

Useful Formulas

- **Cylindrical coordinates**: $(r, \theta, z),\ 0 \le r < \infty,\ 0 \le \theta \le 2\pi,\ -\infty < z < \infty.$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z;\quad dV = r\,dr\,d\theta\,dz$

- **Spherical Coordinates**: $(\rho, \theta, \phi),\ 0 \le \rho < \infty,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

$x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi;\quad dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$

- If $\vec{F}(x,y) = F_1\vec{i} + F_2\vec{j}$, then curl $\vec{F} = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}.$

- **Green's Theorem**: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$

- The flux of $\vec{F}$ through a surface given by the graph of $z = f(x,y)$ is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_R \vec{F}(x,y,f(x,y))\cdot(-f_x,-f_y,1)\ dx\,dy.$$

- The flux of $\vec{F}$ through a cylinder of radius R is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R\cos\theta, R\sin\theta, z)\cdot(\cos\theta,\sin\theta,0)\ R\ dz\,d\theta.$$

- The flux of $\vec{F}$ through a sphere of radius R is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)\cdot(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)\ R^2\sin\phi\,d\phi\,d\theta.$$

- Cartesian Coordinate Definition of **Divergence**: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{div}\ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- Cartesian Coordinate Definition of **Curl**: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{curl}\ \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

- **The Divergence Theorem:** $\displaystyle\int_S \vec{F}\cdot d\vec{A} = \int_W \text{div}\ \vec{F}\ dV.$

- **Stokes' Theorem** $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\ \vec{F}\cdot d\vec{A}.$

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \ (n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$
5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2-a^2}[a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2}[b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2}[b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+-+-+-+\ldots)$ (signs alternate)
15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

    $(-++--++--\ldots)$ (signs alternate in pairs)
16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

    $(++--++--\ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx, \quad n \text{ positive}$
18. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx, \quad n \text{ positive}$
19. $\displaystyle\int \frac{1}{\sin^m x}\ dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2} x\ dx}, \quad m \neq 1, m \text{ positive}$
20. $\displaystyle\int \frac{1}{\sin x}\ dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$
21. $\displaystyle\int \frac{1}{\cos^m x}\ dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2} x\ dx}, \quad m \neq 1, m \text{ positive}$
22. $\displaystyle\int \frac{1}{\cos x}\ dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$
23. $\displaystyle\int \sin^m x \cos^n x\ dx:$

    If n is odd, let $w = \sin x$.
    If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
    If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
    The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
25. $\displaystyle\int \frac{bx + c}{x^2 + a^2}\ dx = \frac{b}{2}\ln|x^2 + a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}(\ln|x-a| - \ln|x-b|) + C, \quad a \neq b$
27. $\displaystyle\int \frac{cx + d}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}, \sqrt{x^2-a^2}, a > 0$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
29. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$
30. $\displaystyle\int \sqrt{a^2 \pm x^2}\ dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\ dx\right) + C$
31. $\displaystyle\int \sqrt{x^2 - a^2}\ dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} + a^2\int \frac{1}{\sqrt{x^2 - a^2}}\ dx\right) + C$