Polytechnic University

MA 2122 FINAL JUNE 2, 2003

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have 90 minutes to answer the following questions. You must show all your work as clearly as possible. You may use a calculator. The last three pages contain a list of useful formulas. You may tear these pages out.

Problem	Possible	Points
1	20	
2	15	
3	10	
4	20	
5	10	
6	15	
7	10	
Total	100	

(1) (Worksheet I, II) In each of the following cases, set up an appropriate iterated integral. (You do not need to evaluate the integrals.)

(a) Let R be the region in the xy-plane bounded by the lines

$$y = -2x + 1, \; y = 2x + 3 \text{ and } y = 1.$$

A double integral, in Cartesian coordinates, representing the area of R is

(b) Reversing the order of integration, the integral $\int_{-1}^{0} \int_{-\pi/2}^{\arcsin y} f(x, y)\, dx\, dy$ becomes

(c) The integral $\int_{0}^{1} \int_{1-x}^{\sqrt{1-x^2}} f(x, y)\, dy\, dx$ converted to polar coordinates is

(d) The integral $\int_{0}^{2\pi} \int_{0}^{3} \int_{0}^{r} r\, dz\, dr\, d\theta$ converted to Cartesian coordinates is

(2) (Worksheet III) Consider the vector field

$$\vec{F}(x, y) = -y\vec{i} + x\vec{j}$$

and let $C = C_1 + C_2 + C_3$ be a piecewise smooth curve where

(a) C_1 is the straight line from $(3, 2)$ to $(1, 0)$,

(b) C_2 is the counterclockwise half circle of radius 1 centered at the origin, starting from $(1, 0)$ and ending at $(-1, 0)$,

(c) C_3, is the portion of the parabola $2y = 2x^2 - 3x - 5$ from the point $(-1, 0)$ to the point $(3, 2)$.

Calculate the line integral of $\vec{F}$ along C.

(3) (Worksheet II) Let W be the upper spherical cap obtained by cutting the sphere of radius 5 centered at the origin with the plane $z = 3$.

Find the total mass of W assuming that its density at each point (x, y, z) is

$$\delta(x, y, z) = z.$$

(Hint: Use cylindrical coordinates.)

(4) Fill in the blanks. (You do not need to show your work.)

(a) Let C be a curve from $(1,0,1)$ to $(2,\pi/6,-1)$ and let $\vec{F}$ be the vector field defined by

$$\vec{F}(x,y,z) = -\frac{\sin y}{x^2z^2}\vec{i} + \frac{\cos y}{xz^2}\vec{j} - 2\frac{\sin y}{xz^3}\vec{k}.$$

Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} =$

(b) The force of gravity on a particle of mass m at a point with position vector $\vec{r}$ is given by

$$\vec{F} = -\frac{GMm\vec{r}}{r^3},$$

where $r = ||\,\vec{r}\,||$, G is the gravitational constant, and M is the mass of Earth. Then the work done by the force of gravity on a particle of mass m as it moves from $8,000$ km to $10,000$ km from the center of Earth is

(c) Let $\vec{F}(x,y,z) = z\vec{i}$.

If S is a square of side 5 in the yz-plane with one corner at the origin, one edge along the positive y-axis, one along the negative z-axis, and oriented in the negative x-direction, then $\displaystyle\int_S \vec{F}\cdot d\vec{A} =$

(d) Suppose that $\vec{F}$ is a vector field of constant divergence, and that S is a sphere of radius 2 with outward orientation.

If $\displaystyle\int_S \vec{F}\cdot d\vec{A} = 12$, then div $\vec{F} =$

(5) (Problem 15, Page 878) Compute the flux of the vector field $\vec{F} = z^2\vec{k}$ through the upper hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geq 0$, oriented away from the origin.

(6) (Worksheet IV) Compute the flux of the vector field $\vec{F}(\vec{r}) = \vec{r}$, through the **closed** solid

$$1 \leq z \leq 10 - \left(x^2 + y^2\right)$$

with outward orientation.

(7) (Worksheet IV) Let C be the circle of radius 3 in the xy-plane, centered at the origin and oriented counterclockwise as viewed from the positive z-axis.

Use Stokes' Theorem to compute the line integral

$$\int_C \left((yz^2 - y)\vec{i} + (xz^2 + x)\vec{j} + x^2\vec{k}\right) \cdot d\vec{r}.$$

Useful Formulas

- **Cylindrical coordinates**: $(r, \theta, z),\ 0 \leq r < \infty,\ 0 \leq \theta \leq 2\pi,\ -\infty < z < \infty.$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z;\ \ dV = r\,dr\,d\theta\,dz$

- **Spherical Coordinates**: $(\rho, \theta, \phi),\ 0 \leq \rho < \infty,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi.$

$x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi;\quad dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$

- If $\vec{F}(x, y) = F_1\vec{i} + F_2\vec{j}$, then curl $\vec{F} = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}.$

- **Green's Theorem**: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$

- The flux of $\vec{F}$ through a surface given by the graph of $z = f(x, y)$ is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y))\cdot(-f_x, -f_y, 1)\ dx\,dy.$$

- The flux of $\vec{F}$ through a cylinder of radius R is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R\cos\theta, R\sin\theta, z)\cdot(\cos\theta, \sin\theta, 0)\ R\ dz\,d\theta.$$

- The flux of $\vec{F}$ through a sphere of radius R is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)\cdot(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\ R^2\sin\phi\,d\phi\,d\theta.$$

- Cartesian Coordinate Definition of **Divergence**: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{div}\ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- Cartesian Coordinate Definition of **Curl**: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{curl}\ \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

- **The Divergence Theorem:** $\displaystyle\int_S \vec{F}\cdot d\vec{A} = \int_W \text{div}\ \vec{F}\ dV.$

- **Stokes' Theorem** $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\ \vec{F}\cdot d\vec{A}.$

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\displaystyle\int x^n \; dx = \frac{1}{n+1} x^{n+1} + C, \; (n \neq -1)$
2. $\displaystyle\int \frac{1}{x} \; dx = \ln|x| + C$
3. $\displaystyle\int a^x \; dx = \tfrac{1}{\ln a} a^x + C$
4. $\displaystyle\int \ln x \; dx = x \ln x - x + C$
5. $\displaystyle\int \sin ax \; dx = -\frac{1}{a} \cos ax + C$
6. $\displaystyle\int \cos ax \; dx = \frac{1}{a} \sin ax + C$
7. $\displaystyle\int \tan ax \; dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\displaystyle\int e^{ax} \sin(bx) \; dx = \frac{1}{a^2+b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + C$
9. $\displaystyle\int e^{ax} \cos(bx) \; dx = \frac{1}{a^2+b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + C$
10. $\displaystyle\int \sin(ax) \sin(bx) \; dx = \frac{1}{b^2-a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\displaystyle\int \cos(ax) \cos(bx) \; dx = \frac{1}{b^2-a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\displaystyle\int \sin(ax) \cos(bx) \; dx = \frac{1}{b^2-a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$,e^x, $\cos x$, and $\sin x$

13. $\displaystyle\int x^n \ln x \; dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\displaystyle\int p(x) e^{ax} \; dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+ - + - + - + \ldots)$ (signs alternate)
15. $\displaystyle\int p(x) \sin ax \; dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

    $(- + + - - + + - - \ldots)$ (signs alternate in pairs)
16. $\displaystyle\int p(x) \cos ax \; dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

    $(+ + - - + + - - \ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\int \sin^n x\; dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\; dx, \quad n \text{ positive}$

18. $\int \cos^n x\; dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\; dx, \quad n \text{ positive}$

19. $\int \frac{1}{\sin^m x}\; dx = -\frac{1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x\; dx}, \quad m \neq 1, m \text{ positive}$

20. $\int \frac{1}{\sin x}\; dx = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C$

21. $\int \frac{1}{\cos^m x}\; dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x\; dx}, \quad m \neq 1, m \text{ positive}$

22. $\int \frac{1}{\cos x}\; dx = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C$

23. $\int \sin^m x \cos^n x\; dx :$

If n is odd, let $w = \sin x$.
If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\int \frac{1}{x^2 + a^2}\; dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

25. $\int \frac{bx + c}{x^2 + a^2}\; dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

26. $\int \frac{1}{(x-a)(x-b)}\; dx = \frac{1}{(a-b)} (\ln |x-a| - \ln |x-b|) + C, \quad a \neq b$

27. $\int \frac{cx + d}{(x-a)(x-b)}\; dx = \frac{1}{(a-b)} [(ac+d) \ln |x-a| - (bc+d) \ln |x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}, a > 0$

28. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

29. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$

30. $\int \sqrt{a^2 \pm x^2}\; dx = \frac{1}{2} \left(x\sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}}\; dx \right) + C$

31. $\int \sqrt{x^2 - a^2}\; dx = \frac{1}{2} \left(x\sqrt{x^2 - a^2} + a^2 \int \frac{1}{\sqrt{x^2 - a^2}}\; dx \right) + C$