

# Polytechnic University

MA 2112

FINAL

JUNE 2, 2003

Print Name:

Signature:

ID #:

Instructor/Section:

**Directions:** You have 90 minutes to answer the following questions. You must show all your work as clearly as possible. You may use a calculator. The last page contains a list of useful formulas. You may tear that page out.

Problem	Possible	Points
1	15	
2	20	
3	10	
4	15	
5	15	
6	15	
7	10	
Total	100	

(1) Determine whether each of the following statements is TRUE or FALSE. (You do not need to show your work.)

(a) If  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are nonzero vectors, and  $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ , then  $\vec{v} = \vec{w}$ .

(b) The triangle in 3-space with vertices  $(3, 1, 0)$ ,  $(0, 1, 0)$  and  $(1, 0, -1)$  has a right angle.

(c) If  $f(x, y)$  has a local maximum at  $(x_0, y_0)$ , then so does the function  $g(x, y) = \lambda f(x, y)$  for any real number  $\lambda$ .

(d) The function  $f(x, y) = e^{xy^2}$  increases in the  $y$ -direction at the point  $(-1, -2)$ .

(e) At the point  $(3, 0)$ , the function  $g(x, y) = x^2 + y^2$  has the same maximal rate of increase as that of the function  $h(x, y) = 2xy$ .

(2) Fill in the blanks. (You do not need to show your work.)

(a) Let  $f(x, y)$  be a function and let  $H(u, v) = f(u \cos(v), u \sin(v))$ .

If  $f_x(0, 1) = -2 = f_y(1, 0)$ ,  $f_y(0, 1) = 4 = f_x(1, 0)$ , then

$$H_u(1, \pi/2) = \underline{\hspace{2cm}}$$

(b) If the Taylor polynomial of degree 2 for  $f(x, y) = (\sin x)(\cos y)$  at the point  $(a, b)$  is

$$Q(x, y) = \frac{\pi}{2}x - xy,$$

then

$$a = \underline{\hspace{2cm}} \quad b = \underline{\hspace{2cm}}$$

(c) The directional derivative of the function  $f(x, y, z) = 3x^2y^2z$  at  $(-1, 1, 1)$  in the direction of  $2\vec{i} + 2\vec{j} - \vec{k}$  is

$$\underline{\hspace{2cm}}$$

(d)  $f(x, y) = 8xy - \frac{1}{4}(x + y)^4$  has a saddle point at  $(x_0, y_0) = \underline{\hspace{2cm}}$

- (3) (Problem 10, Page 735) Assume that two products are manufactured in quantities  $q_1$  and  $q_2$  and sold at prices per unit of  $p_1$  and  $p_2$  respectively, and that the cost of producing them is given by

$$C = 2q_1^2 + 2q_2^2 + 10.$$

- (a) Find the maximum **profit** assuming that the prices are fixed.  
(b) Find the rate of change of the maximum profit as  $p_1$  increases.

(4) (Worksheet III) The surface  $S$  is represented by the equation  $F = 0$  where

$$F(x, y, z) = x^4 + y^4 + z^4 - 3.$$

Let  $A$  and  $B$  be the two points on  $S$  at which the tangent plane is parallel to the plane

$$x - y + z = 5.$$

Find an equation of the plane passing through the points  $A$ ,  $B$  and  $(1, 1, 1)$ .

- (5) (Practice Exam) Let  $L$  be the tangent plane to the surface  $z = 1 - x^3y^3$  at the point  $(1, -1, 2)$ . Find two perpendicular **unit** vectors on  $L$ .

- (6) (Worksheet IV) Let  $L(x, y)$  be the Taylor polynomial of degree 1 about  $(0, 0)$  for the function

$$f(x, y) = \ln(1 + x^2 - y).$$

What is the shortest distance from the plane  $z = L(x, y)$  to the point  $(0, 1, 1)$ ?

- (7) (Problem 4, Page 725) Use Lagrange multipliers to find the maximum and minimum value(s), if they exist, of

$$f(x, y, z) = 2x + y + 4z$$

subject to the constraint  $x^2 + y^2 + z^2 = 16$ .

*(Classify the value(s) found as a minimum or a maximum.)*

## Useful Formulas

- The **dot product** of two vectors:

$$\vec{v} \cdot \vec{w} = \|\vec{u}\| \|\vec{w}\| \cos \theta = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

- The **cross product** of two vectors:

$$\vec{v} \times \vec{w} = (\|\vec{u}\| \|\vec{w}\| \sin \theta) \vec{n} = (v_2 w_3 - v_3 w_2) \vec{i} + (v_3 w_1 - v_1 w_3) \vec{j} + (v_1 w_2 - v_2 w_1) \vec{k}.$$

- **Volume of a parallelepiped** with edges  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ : Volume =  $|\vec{b} \times \vec{c} \cdot \vec{a}|$ .

- **Chain Rule** If  $f$ ,  $g$  and  $h$  are differentiable and if  $z = f(x, y)$ , and  $x = g(t)$ , and  $y = h(t)$ , then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

If  $f$ ,  $g$ ,  $h$  are differentiable and if  $z = f(x, y)$ , and  $x = g(u, v)$ , and  $y = h(u, v)$ , then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.$$

- **Taylor polynomial** of degree 2 approximating  $f(x, y)$  for  $(x, y)$  near  $(a, b)$ :

$$\begin{aligned} f(x, y) \approx Q(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2. \end{aligned}$$

- **Second derivative test for functions of two variables:**

Suppose  $\nabla f(x_0, y_0) = \vec{0}$ . Let  $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$ .

- (1) If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $f$  has a local minimum at  $(x_0, y_0)$ .
- (2) If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $f$  has a local maximum at  $(x_0, y_0)$ .
- (3) If  $D < 0$ , then  $f$  has a saddle point at  $(x_0, y_0)$ .
- (4) If  $D = 0$  the test is inconclusive.