Directions: You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator. When you use the Gaussian algorithm explain each step by a remark like “adding $-4$ times the first row to the third row” or $R_3 \rightarrow R_3 - 4R_1$. If you use a computation rule, like $(AB)^T = B^T A^T$, state the rule itself.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
(1) Determine whether each of the following statements is True or False. You do not have to explain.

(a) Matrices $A$ and $A^T$ have the same nullity.

(b) Any 4 vectors of $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$.

(c) A skew-symmetric matrix must have an eigenvalue of 0.

(d) Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be non-zero vectors in $\mathbb{R}^n$. If $m < n$, then the vectors are linearly independent.

(e) If the characteristic polynomial of a square matrix $A$ is $p(\lambda) = (\lambda - 2)^2(\lambda - 5)$, then $\text{tr}(A) = 9$. 

(2) Fill up in blanks. You do not have to explain.

(a) If \( \begin{pmatrix} 3 \\ x \\ -2 \end{pmatrix} \) is in the null space of \( A = \begin{pmatrix} 4 & 3 & -3 \\ 2 & -1 & y \\ 0 & 0 & 0 \end{pmatrix} \) then

\[
x = \underline{\phantom{0}},
\]
and

\[
y = \underline{\phantom{0}}.
\]

(b) If \( A \) is an \( 5 \times 5 \) matrix such that \( \text{tr}(A) = -3 \), and \( \det(A) = 2 \), then

\[
\det(-2 \text{tr}(A)A) = \underline{\phantom{0}}
\]

\[
\text{tr}(-3 \det(A)A) = \underline{\phantom{0}}.
\]

(c) The basis for the vector space

\[
V = \left\{ \begin{pmatrix} 0 \\ x \\ -x \\ 0 \end{pmatrix} ; x \in \mathbb{R} \right\}
\]

is \underline{\phantom{0}}.

(d) Let \( A = \begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix} \). The \( f(x) = x^2 - 2 \) is \underline{\phantom{0}}.
(3) Fill in the blanks. You do not have to give explanations.

A row-echelon form of

\[ A = \begin{pmatrix}
1 & 2 & 0 & -3 & 1 & 0 \\
1 & 2 & 1 & -3 & 1 & 2 \\
1 & 2 & 0 & -3 & 2 & 1 \\
3 & 6 & 1 & -9 & 4 & 3
\end{pmatrix} \]

\[ \text{ref}(A) = \begin{pmatrix}
1 & 2 & 0 & -3 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

(a) A basis for the row space of \( A \) is:

(b) A basis for the column space of \( A \) is:

(c) A basis for the null space of \( A \) is:

(d) The rank of \( A \) is: ________

(e) The nullity of \( A \) is: ________
(4) In each part you have to determine whether $W$ is a subspace of $V$. You must show all your explanations. In each part you must clearly state the “zero-vector” in $V$.

(a) Let $V$ be the vector space of all $2 \times 2$ real matrices, and $W$ the set of all $2 \times 2$ matrices with trace zero. Is $W$ a subspace of $V$?

(b) Let $V$ be the vector space of all vectors in $\mathbb{R}^4$ and define $W$ by

$$W = \{(x, x^2, x^3, x^4) : x \in \mathbb{R}\}.$$ 

Is $W$ a subspace of $V$?
(5) Write down the standard basis and the dimension of each of the following vector spaces over the real field. You do not need to show work.

(a) The space consisting of all linear transformations

\[ T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ x + y \\ y - z \\ x \end{pmatrix}. \]

(b) The space consisting of all 3 × 3 lower triangular matrices.
(6) Let

\[ A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \]

(a) Find the characteristic equation of \( A \). (Show your work.)

(b) Find the eigenvalues and the corresponding eigenvectors of the matrix \( A \). (Show your work.)
(Continued from the previous page)
(c) Is $A$ diagonalizable? If yes, find $D$, $P$, and $P^{-1}$ such that $A = PDP^{-1}$ (Show your work).
(7) Let $T : \mathbf{V} \rightarrow \mathbf{W}$ be the transformation defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 2y \\ y + 2z \end{pmatrix}.$$ 

You must show your work.

(a) Show that $T$ is a linear transformation.

(b) Find the standard matrix representing $T$. 
(c) Show that $T$ is invertible.

(d) Find a vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$, where $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $\vec{w} \in W$. 