Directions: You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator. When you use the Gaussian algorithm explain each step by a remark like “adding $-4$ times the first row to the third row” or $R_3 \rightarrow R_3 - 4R_1$. If you use a computation rule, like $(AB)^T = B^T A^T$, state the rule itself.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

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(1) Determine whether each of the following statements is True or False. (You do not have to explain.)

(a) If \( \lambda = 0 \) is an eigenvalue of a matrix \( A \), then \( A \) is not diagonalizable.

(b) If a row-echelon form of the augmented matrix of a linear system \( B\vec{x} = \vec{b} \) has a full row of zeros, then the system has infinitely many solutions \( \vec{x} \).

(c) The set of vectors
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ \end{pmatrix} \right\}
\]
forms a basis for \( \mathbb{R}^3 \).

(d) If \( C \) is an \( 8 \times 4 \) matrix with rank \( (C) = 3 \), then nullity \( (C^T) = 5 \).

(e) The set of vectors
\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \end{pmatrix} \right\}
\]
is linearly independent.
(2) Find the dimension and a basis for the solution space of the homogeneous linear system

\[ \begin{align*}
    x + 2y - 3z + 2s - 4t &= 0 \\
    2x + 4y - 5z + s - 6t &= 0 \\
    5x + 10y - 13z + 4s - 16t &= 0 \\
    8x + 16y - 21z + 7s - 26t &= 0.
\end{align*} \]
(3) Fill in the blanks. (You do not have to give explanations.)

(a) Consider the following four transformations.

\[ S : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad T(x_1, x_2, x_3) = (4x_1 + 6x_2 + 4x_3, 2x_1 + 3x_2, 7x_3), \]
\[ T : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad T(x_1, x_2, x_3) = (x_1 - 4x_3, -3x_2 + 2x_3, 2x_3), \]
\[ U : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}, \]
\[ V : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 5x_2 \\ x_2 - 2x_3 \end{pmatrix}. \]

(i) The standard matrix of \( S \) is

(ii) Which of the four transformations is/are linear? 

(b) Let

\[ C = \begin{bmatrix}
1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}. \]

(i) The \( \det(C) = \) 

(ii) Is \( C \) singular or nonsingular?
(4) Fill in the blanks. (You do not have to give explanations.)

A row-echelon form of

\[ A = \begin{pmatrix} 2 & 6 & 0 & 2 & 8 \\ 5 & 15 & 2 & 1 & 24 \\ -10 & -30 & 0 & -10 & -39 \\ -3 & -9 & 1 & -5 & -10 \end{pmatrix} \]

is

\[ \text{ref}(A) = \begin{pmatrix} 1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

(a) A basis for the row space of \( A \) is:

(b) A basis for the column space of \( A \) is:

(c) The rank of \( A \) is: 

(d) The nullity of \( A^T \) is:
(5) In each part you have to determine whether $W$ is a subspace of $V$. You must show all your explanations. In each part you must clearly state the “zero-vector” in $V$.

(a) Recall that a square matrix $A$ is called skew-symmetric, if $A^T = -A$. Let $V$ be the vector space of all $3 \times 3$ real matrices, and $W$ the set of all $3 \times 3$ skew-symmetric real matrices. Is $W$ a subspace of $V$?

(b) Let $V$ be the vector space of all vectors in $\mathbb{R}^2$ and define $W$ by

\[ W = \{ (x, y) \in \mathbb{R}^2 : x - 2y = 3 \}. \]

Is $W$ a subspace of $V$?
(6) Let $A$ be a $3 \times 3$ matrix upper triangular matrix with $\text{tr}(A) = 9$, $\det(A) = 24$ and one eigenvalue $\lambda_1 = 2$. Answer the following questions. You must show your work.

(a) What are the other 2 eigenvalues?

(b) What is the characteristic polynomial of $A$?

(c) What is one possibly for the $3 \times 3$ matrix $A$?
(7) (Continued on the next page) Let

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 8 & 10 \\
0 & -1 & 2
\end{pmatrix}.
\]

(a) Find the eigenvalues of the matrix \( A \). (Show your work.)

(b) Find the eigenvectors of the matrix \( A \). (Show your work.)

(c) Is it possible to find a matrix \( P \) such that \( AP = PD \)?
(d) (Continued from the previous page) Suppose \( f(x) = x^2 + 3 \). Compute \( f(A) \).
(Show your work.)

(e) Find the eigenvalues and eigenvectors of the matrix \( f(A) \). (Show your work.)