Directions: You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator.

When you use the Gaussian algorithm explain each step by a remark like “adding $-4$ times the first row to the third row” or $R_3 = R_3 - 4R_1$.

If you use a computation rule, like $(AB)^T = B^T A^T$, state the rule itself.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

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(1) Determine whether each of the following statements is True or False. You do not have to explain.

(a) If $M$ is an invertible $n \times n$ matrix, then the linear system $M\vec{x} = \vec{b}$ has a unique solution for every vector $\vec{b} \in \mathbb{C}^n$.

(b) If $P$ and $Q$ are two $n \times n$ invertible matrices, then $P^{-1}(P^TQ)^T = Q^T$.

(c) If the characteristic polynomial of a square matrix $C$ is $p(\lambda) = (16 - \lambda^2)(\lambda + 5)$, then the matrix $A$ must be diagonalizable.

(d) If $A$ and $B$ are two $n \times n$ matrices such that $AB = 0$, (where 0 is the $n \times n$ zero-matrix), then either $A = 0$ or $B = 0$.

(e) Any 4 vectors of $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$. 
(2) Fill in the blanks. You do not have to explain.

(a) If \( p(\lambda) = (\lambda - 3)^2(\lambda + 5)(\lambda + 7) \) is the characteristic polynomial of a matrix \( A \), then the size of the matrix \( A \) is ____________.

(b) If \( A \) is a \( 7 \times 11 \) matrix with \( \text{rank}(A) = 6 \), then \( \text{rank}(A^T) = \) ____________, nullity\((A) = \) ____________, and nullity\((A^T) = \) ____________.

(c) The vector \( \vec{v} = \begin{pmatrix} -5 + i \\ -1 + i \\ 2 \end{pmatrix} \) is an eigenvector of \( \begin{pmatrix} -2 & -2 & 9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix} \) with corresponding eigenvalue \( \lambda = \) ____________.
   (\text{Hint: Recall the definition of eigenvector: } A\vec{v} = \lambda \vec{v}.)

(d) Let \( A \) be a \( 4 \times 4 \) matrix with \( \det(A) = -2 \) and \( \text{tr}(A) = 3 \), and let \( B \) be a \( 4 \times 4 \) matrix with \( \det(B) = 16 \) and \( \text{tr}(B) = 5 \). Evaluate each of the following expressions, or state if there is not enough information to evaluate.

\[
\det(2A^2B^{-1}) = \text{__________________________}
\]

\[
\text{tr}(2B^T - 3I) = \text{__________________________}
\]

(e) If \( A \) is a \( 4 \times 4 \) matrix over \( \mathbb{C} \) with \( \text{rank}(A) = 4 \), then the number of solutions to the homogenous system \( A\vec{x} = \vec{0} \) over \( \mathbb{C} \) is ____________.
Let \( A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -3 & -6 & 1 \end{pmatrix} \).

(a) Is the vector \( \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \) in the null space of \( A \)?

(b) Find the null space of \( A \) (i.e. the solution space of \( A\vec{x} = \vec{0} \)). Use the correct set notation to give your answer.

(c) Find a basis for the null space of \( A \).

(d) Find a basis for the column space of \( A \).

(e) What is the dimension of the column space of \( A \)?

(f) What is the dimension of the null space of \( A \)?
(4) Answer the following questions. You do not need to explain.

(a) Consider the following four sets of vectors from \( \mathbb{R}^3 \):

\[
\mathcal{P} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -15 \end{pmatrix} \right\}, \quad \mathcal{R} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -18 \end{pmatrix} \right\}, \\
\mathcal{S} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \right\}, \quad \mathcal{T} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} \right\}.
\]

Which of these four sets form a linearly independent set? ____________

Which of these four sets form a basis for \( \mathbb{R}^3 \)? ____________

(b) Let \( \mathcal{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a linear transformation defined by \( \mathcal{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_2 \end{pmatrix} \).

The matrix representation of \( \mathcal{T} \) with respect to the standard basis of \( \mathbb{R}^3 \) is ____________

(c) Which of the following sets form a subspace for \( \mathcal{M}_{2\times2} \), the vector space of all \( 2 \times 2 \) matrices over \( \mathbb{R} \)? Circle all subspaces of \( \mathcal{M}_{2\times2} \).

(i) The set of all \( 2 \times 2 \) diagonal matrices.

(ii) The set of all \( 2 \times 2 \) triangular matrices.

(iii) The set of all \( 2 \times 2 \) matrices \( A \) such that \( a_{12} = 1 \).

(iv) The set of all \( 2 \times 2 \) matrices \( B \) such that \( b_{11} = 0 \).

(v) The set of all \( 2 \times 2 \) singular matrices.
(5) Short proof. **Do only ONE of the two parts.** Cross out the part you do not want to do. No extra credit for doing both parts. If you do both parts, you will get credit for part (a) only.

(a) Let $V$ be the vector space of all continuous functions defined on the interval $[-5, 5]$, and let

$$W = \{ f \in V : f(-t) = -f(t) \}.$$

Determine whether $W$ is a subspace of $V$. You must clearly show your explanation. You must clearly state the “zero-vector” in $V$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the transformation defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 \\ x_3 \\ x_1 \\ x_2 + x_1 \end{pmatrix}.$$

*Using the definition*, determine whether the transformation $T$ is linear. Clearly show your explanation.
(6) Let

\[ A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}. \]

(a) Find the eigenvalues of \( A \).

(b) For each of the eigenvalues of \( A \) find the corresponding eigenspace, and give a basis for each eigenspace.
(7) Show your work.

(a) Let the vectorspace $\mathcal{V}$ be given by

$$\mathcal{V} = \left\{ \begin{pmatrix} a \\ a - 2c \\ b \\ b + 5c \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

(i) What is the dimension of $\mathcal{V}$?

(ii) Find a basis for the vectorspace $\mathcal{V}$.

(b) Let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}, \quad \text{and} \quad \vec{u} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}.$$ 

The vector $\vec{u}$ \underline{is not} in the subspace spanned by the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$. If it is, then find the coordinate vector with respect to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.