Polytechnic University

MA 1112/1412 Sample Exam 2 Spring 2005

(1) (Page 277, Problems 1–10; Page 481, Problems 3–15) Fill in the blanks. You do not need to explain.

(a) The function $y(x) = Ae^x - 1$ is a solution of the initial value problem $y' = y + 1$, $y(0) = 5$ if $A =$____________.

(b) The function $y(x) = 10e^{-x} + x + B$ is a solution of the differential equation $y' = x - y$ if $B =$____________.

(c) The function $y(t) = 2\cos(3t)$ is a solution of the differential equation $y'' + ky = 0$ if $k =$____________.

(d) The general solution of the differential equation $\frac{dr}{dp} = 3\sin p$ is ________________________.

(2) (Page 284, Problems 5 and 6) On the moon the acceleration due to gravity is 5 ft/sec^2. A ball is tossed vertically upward with an initial velocity of 10 ft/sec from a cliff and it hits the ground below the cliff in 5 seconds. How high is the cliff? You must show all your work.

(3) (Page 296, Problems 2–47) Circle the correct choice. You need not show any work.

(a) $\int \sin^5(2x)\cos(2x)\,dx =$

(i) $\frac{\sin^6(2x)}{12} + C$

(ii) $\frac{\sin^6(2x)}{6} + C$

(iii) $\frac{\sin^6(2x)}{3} + C$

(iv) $\frac{\cos^5(2x)}{3} + C$

(v) $\frac{\cos^5(2x)}{6} + C$

(b) $\int x\sqrt{x+3}\,dx =$

(i) $\frac{2}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C$

(ii) $\frac{2(x+3)^{\frac{3}{2}}}{3} + C$

(iii) $\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C$

(iv) $\frac{3(x+3)^{\frac{3}{2}}}{2} + C$

(v) $\frac{4x^2(x+3)^{\frac{3}{2}}}{3} + C$

(4) (Page 298, Problem 72, Page 304, Problem 50) Let f be a twice differentiable function. Some of the values of f and its derivative are given in the table below.

x	$f(x)$	$f'(x)$
0	1	3
4	5	2
16	7	1

(a) Evaluate the integral $\int_0^{16} \frac{f'(\sqrt{x})}{\sqrt{x}}\, dx$. Show all your work.

(b) Evaluate the integral $\int_4^{16} (16 - 2x) f''(x)\, dx$. Show all your work.

(5) (Page 308, Problems 1–35) Find each of the following indefinite integrals using the table of integrals. You must show all your work and state the rule(s) from the table that you are using.

(a) $\displaystyle\int e^{5x} \cos(3x)\, dx$

(b) $\displaystyle\int \frac{1}{t^2 + 4t + 6}\, dt$

(6) (Page 303, Problems 1–35) Find the indefinite integrals. You must show work.

(a) $\int 1 - \arctan(x)\, dx$

(b) $\int u^5 \ln(5u)\, du$

(c) $\int \frac{2x + 3}{x(x + 2)(x - 1)}\, dx$

(7) (Page 321, Problems 17, 18) A tank contains 1000 liters of water. Let $r(t)$ represent the rate in liters per hour at which water is leaking out from the tank after t hours. The table below gives some values of $r(t)$. Use the midpoint rule to estimate the total amount of water ***remaining*** in the tank after eight hours.

t hours	0	1	2	3	4	5	6	7	8
$r(t)$ liters/hr	6	5.9	5.7	5.4	5	4.5	4	3.5	3

Indicate clearly the number of subdivisions and show all your work.

(8) (Page 321, Problem 10) Left, right, midpoint, trapeziod rules used to estimate $\int_0^{1/2} \frac{1}{1+x^2}\, dx$. Determine which of the following must be TRUE. You do not have to explain.

(a) LEFT(1000) $\leq$ RIGHT(1000)

(b) $\int_0^{1/2} \frac{1}{1+x^2}\, dx \leq$ LEFT(1000)

(c) TRAP(1000) $\leq$ MID(1000)

(d) LEFT(1000) $\leq$ MID(1000)

(e) RIGHT(1000) $\leq$ RIGHT(5000)

(9) (Page 277, Problem 13) Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$ measured in weeks, with the result that the fish die at a rate given by

$$\frac{dP}{dt} = \frac{k}{(t+4)^2},$$

where k is a nonzero constant. If there were initially 1000 fish in the lake and 500 left after 2 weeks, how long did it take all the fish in the lake to die? You must show all your work.

(10) (Page 304, Problem 51; Page 297, Problem 69) Find the exact value(s) of a such that the average value of the function

$$f(x) = xe^{2x}$$

between $x = 0$ and $x = a$ is $\dfrac{1}{4a}$.

Show your work.

Useful formulas

- *Fundamental Theorem of Calculus*: If F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$
- *The average value* of a function f on an interval $[a,b]$ is equal to $\frac{1}{b-a}\int_a^b f(x)\,dx$
- *Comparison of Definite Integrals*: If f is continuous and $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
- The *acceleration* due to gravity, g:

$$g = 9.81\text{m/sec}^2, \qquad \text{or} \qquad g = 32\text{ft/sec}^2$$

- *Integration by Parts*:

$$\int u dv = uv - \int v du \qquad \text{or} \qquad \int uv' dx = uv - \int vu' dx$$

- *Numerical Approximations:*

$$\text{TRAP}(n) = \frac{\text{LEFT}(n)+\text{RIGHT}(n)}{2}$$

- **Differentiation formulas**

$\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\sin(x)) = \cos x$ $\frac{d}{dx}(\tan(x)) = \sec^2 x$ $\frac{d}{dx}(\sec(x)) = \sec x \tan x$	$\frac{d}{dx}(a^x) = (\ln a)a^x$ $\frac{d}{dx}(\cos(x)) = -\sin x$ $\frac{d}{dx}(\cot(x)) = -\csc^2 x$ $\frac{d}{dx}(\csc(x)) = -\csc x \cot x$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

Your signature:

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C,\ (n \neq -1)$
2. $\int \frac{1}{x}\ dx = \ln|x| + C$
3. $\int a^x\ dx = \frac{1}{\ln a}a^x + C$
4. $\int \ln x\ dx = x\ln x - x + C$
5. $\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
6. $\int \cos ax\ dx = \frac{1}{a}\sin ax + C$
7. $\int \tan ax\ dx = -\frac{1}{a}\ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax}\sin(bx)\ dx = \frac{1}{a^2+b^2}e^{ax}[a\sin(bx) - b\cos(bx)] + C$
9. $\int e^{ax}\cos(bx)\ dx = \frac{1}{a^2+b^2}e^{ax}[a\cos(bx) + b\sin(bx)] + C$
10. $\int \sin(ax)\sin(bx)\ dx = \frac{1}{b^2-a^2}[a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax)\cos(bx)\ dx = \frac{1}{b^2-a^2}[b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax)\cos(bx)\ dx = \frac{1}{b^2-a^2}[b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x\ dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x)e^{ax}\ dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \cdots + C$

    $(+-+-+-+\ldots)$ (signs alternate)
15. $\int p(x)\sin ax\ dx = -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a^2}p'(x)\sin(ax) + \frac{1}{a^3}p''(x)\cos(ax) - \cdots + C$

    $(-++--++--\ldots)$ (signs alternate in pairs)
16. $\int p(x)\cos ax\ dx = \frac{1}{a}p(x)\sin(ax) + \frac{1}{a^2}p'(x)\cos(ax) - \frac{1}{a^3}p''(x)\sin(ax) - \cdots + C$

    $(++--++--\ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad n \text{ positive}$
18. $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad n \text{ positive}$
19. $\int \frac{1}{\sin^m x} \, dx = -\frac{1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$
20. $\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C$
21. $\int \frac{1}{\cos^m x} \, dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$
22. $\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C$
23. $\int \sin^m x \cos^n x \, dx :$
    If n is odd, let $w = \sin x$.
    If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
    If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
    The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, \; a \neq 0$
25. $\int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan\left(\frac{x}{a}\right) + C, \; a \neq 0$
26. $\int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} (\ln |x-a| - \ln |x-b|) + C, \quad a \neq b$
27. $\int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} [(ac+d) \ln |x-a| - (bc+d) \ln |x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}, a > 0$

28. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
29. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$
30. $\int \sqrt{a^2 \pm x^2} \, dx = \frac{1}{2} \left(x\sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} \, dx \right) + C$
31. $\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left(x\sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C$